

Power Spectra in Spacetime Noncommutative Inflation

Qing-Guo Huang and Miao Li
Institute of Theoretical Physics
Academia Sinica, P. O. Box 2735
Beijing 100080

huangqg@itp.ac.cn
mli@itp.ac.cn

The spacetime uncertainty relation, which deviates from general relativity, emerges in String/M theory. It is possible to observe this deviation through cosmological experiments, in particular through the measurements of CMB power spectrum. This paper extends some previous observations to more general inflation schemes. We find that the noncommutative spacetime effects always suppress the power spectrum of both the scalar and tensor perturbations, and may provide a large enough running of the spectral index to fit the results of WMAP in the inflation model.

Einstein's general relativity is broken by the quantum effects at very short distances. If inflation happened before the hot big bang, it may be possible to observe these quantum effects through cosmological experiments, such as WMAP and SDSS, since the short distances are stretched to the cosmic ones by the accelerating expansion during inflation. If this is the case, it appears that we need to well understand quantum gravity before we can calculate the power spectrum of cosmological fluctuations accurately. Since String/M theory is the most attractive framework of quantum gravity, it is imperative to investigate the modification of the primordial fluctuation spectrum in string theory. However, there is no fundamental formulation of string theory in a time-dependent background yet. We need resort to some general intuitions gained from studying string theory. Spacetime uncertainty is one of such intuitions and we shall in this paper extend some previous observations concerning the effects caused by spacetime uncertainty in the CMB power spectrum to a more general situation.

The first year results of WMAP [1] put forward more restrictive constraints on cosmological models and confirm the emerging standard model of cosmology, a flat Λ -dominated universe seeded by a nearly scale-invariant adiabatic Gaussian fluctuations. WMAP also brings about some new intriguing results, such as a running spectral index of the scalar fluctuations and an anomalously low quadrupole of CMB angular power spectrum [2]. According to [3-5], spacetime noncommutative power-law inflation naturally produces a large enough running of the spectral index to match the results of WMAP. However the large suppression of C_2^{TT} remains mysterious.

It is generally accepted that the large scale structures of the universe originated from some small seed perturbations [6], which over time grew to become all of the structures we observe today. Quantum fluctuations of the inflaton field are excited during inflation and stretched to cosmological scales with the expansion of our universe. At the same time, being the inflaton fluctuations, ripples in spacetime are also excited and stretched to cosmological scales. The physical wavelengths of these fluctuations are so short that they are sensitive to the physics at the very short distance during the period of inflation. Many authors have discussed how the spectra of fluctuations in inflationary cosmology depend on trans-Planckian physics and whether these trans-Planckian effects can be observed. In general, there are two ways to investigate them. One is that we can take the trans-Planckian physics into account by modifying the dispersion relation [7]. The other is called the minimal trans-Planckian physics [8], in which we calculate the power spectrum of the fluctuations in inflationary cosmology, starting with initial conditions imposed on mode by mode when

the physical wavelength of the fluctuation equals some critical length l_n corresponding to a new energy scale $M_n = l_n^{-1}$. In this model, a mode of fluctuation is created when its wavelength became equal to l_n , while the evolution equations of fluctuation modes are unmodified. Thus the change is entirely encoded in the initial conditions and there is no need to postulate some ad-hoc trans-Planckian physics. As a result, the corrections can be generally expanded in terms of some powers of H/M_n , where H is the Hubble parameter during inflation. Thus we can probe new physics in this way only if the Hubble parameter H during inflation is not much smaller than the new physical scale M_n .

In perturbative string theory, the fundamental degree of freedom is fundamental string with length scale l_s as the minimal physical length scale, implying a stringy uncertainty relation [9]: $\Delta x_p \Delta p \geq 1 + l_s^2 \Delta p^2$ or $\Delta x_p \geq l_s$. Heuristically the minimal distance we can measure by using fundamental string must not be shorter than its length l_s since the fundamental string is extensive. In nonperturbative string theory or M theory, new degrees of freedom such as D-branes and black holes must be taken into account, their effects are suppressed by a factor such as $\exp(-1/g_s)$ ($g_s \rightarrow 0$ in perturbative string theory), where g_s is the string coupling. A D-brane probe with a sufficiently small velocity can be used to probe distances shorter than the string scale. In any physical process a new uncertainty relation [10],

$$\Delta t_p \Delta x_p \geq l_s^2, \quad (1)$$

is observed, where t_p and x_p are the physical time and space. We believe that M theory should not favor a “hard” cut-off, but an uncertainty relation between space and time. However it is hard to incorporate this relation directly in string theory in a fundamental formulation. On a general ground, if inflation is affected by physics at a scale close to string scale or a related scale, one expects that spacetime uncertainty must leave traces in the CMB power spectrum [11,12] (see also [3-5]). Indeed, an explicit calculation based on the model proposed in [12] shows that the effects can be observed [3].

Before we discuss the noncommutative inflation in detail, let us sketch the physics behind this scenario. As in the usual inflation model, we assume that the universe went through a slow-roll period during which cosmic perturbations were generated. The larger the scale of the perturbation, the earlier its generating time. The long wave-length perturbations were generated earlier and crossed out the horizon earlier, and re-entered the horizon later. For an earlier generating time, the spacetime uncertainty relation tells us that the uncertainty in creating time is smaller. Nevertheless, the ratio of the time uncertainty and the creation time is larger, so we expect that the correction of the power

spectrum is larger for a smaller wave-number. In addition, the usual power-law of the spectrum tells us that a smaller wave-number comes with a smaller power, and due the the larger correction of the relative generating time, we expect that the noncommutative correction makes the power of a smaller wave-number even smaller, thus induces a larger spectral index. In the UV end, the story is reversed, so the spectral index for a larger wave-number is smaller, smaller than 1 when the wave-number is larger than a “critical value”.

The spacetime noncommutative effects can be encoded in a new product, start product, replacing the usual algebra product. The evolution of a homogeneous background will not change and the standard cosmological equation of the inflation based on Friedmann-Robertson-Walker (FRW) metric remains the same:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (2)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3M_p^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \quad (3)$$

here M_p is the reduced Planck mass and we assume the universe be spatially flat and the inflaton ϕ be spatially homogeneous. If $\dot{\phi}^2 \ll V(\phi)$ and $\ddot{\phi} \ll 3H\dot{\phi}$, the scalar field shall slowly roll down its potential. Define some slow-roll parameters,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{M_p^2}{2} \left(\frac{V'}{V}\right)^2, \quad (4)$$

$$\eta = \epsilon - \frac{\ddot{H}}{2H\dot{H}} = M_p^2 \frac{V''}{V}, \quad (5)$$

$$\xi^2 = 7\epsilon\eta - 5\epsilon^2 - 2\eta^2 + \zeta^2 = M_p^4 \frac{V'V'''}{V^2}, \quad (6)$$

where $\zeta^2 = (d^3H/dt^3)/(2H^2\dot{H})$, then the slow-roll condition can be expressed as $\epsilon, \eta \ll 1$.

As impressive successes, inflation model not only easily solves the flatness problem, the entropy problem and the horizon problem in hot big bang model, but also provides a reasonable primordial cosmological fluctuations. In order for structure formation to occur via gravitational instability, there must have been small preexisting fluctuations on physical length scales when they crossed inside the Hubble radius in the radiation-dominated and matter-dominated eras. We will consider small perturbations away from the homogeneous and isotropic reference spacetime, the spatially flat Friedmann-Robertson-Walker (FRW) spacetime in our case. These metric perturbations can be decomposed into

different spin modes. The key issue is that general relativity is a gauge theory where the gauge transformations are general coordinate transformations from a local reference frame to another, thus we need to define some gauge invariant quantities, such as the tensor fluctuation h_{ij} and the comoving curvature perturbation \mathcal{R} , to describe the cosmological density perturbations.

We will focus on the scalar perturbation in the spacetime noncommutative inflation first. In the conformal coordinates, the linear scalar perturbations of the metric can be expressed most generally by two scalar degrees of freedom A and ψ and the line-element becomes

$$ds^2 = a^2[(1 + 2A)d\chi^2 - (1 - 2\psi)\delta_{ij}dx^i dx^j], \quad (7)$$

where the conformal time χ is defined as

$$\chi = \int a^{-1} dt. \quad (8)$$

Since the stress tensor does not have any non-diagonal component, we have $\psi = A$. We define the comoving curvature perturbation as

$$\mathcal{R} = \psi + H \frac{\delta\phi}{\dot{\phi}}, \quad (9)$$

which is gauge invariant, where $\delta\phi$ is the quantum fluctuation of inflaton.

For convenience we introduce another time coordinate τ in the noncommutative space-time such that the metric becomes

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 = a^{-2}(\tau)d\tau^2 - a^2(\tau)d\vec{x}^2. \quad (10)$$

Now the uncertainty relation (1) becomes

$$\Delta\tau \Delta x \geq l_s^2, \quad (11)$$

where x is the comoving spatial coordinate. Skipping the detailed discussions in [12], we simply write down the action of the perturbation which incorporates the noncommutative case in four dimensions

$$S = V \int_{k < k_0} d\tilde{\eta} d^3k \frac{1}{2} z_k^2(\tilde{\eta}) (u'_{-k} u'_k - k^2 u_{-k} u_k), \quad (12)$$

where

$$\begin{aligned} z_k^2(\tilde{\eta}) &= z^2 y_k^2(\tilde{\eta}), \quad y_k^2 = (\beta_k^+ \beta_k^-)^{\frac{1}{2}}, \\ \frac{d\tilde{\eta}}{d\tau} &= \left(\frac{\beta_k^-}{\beta_k^+} \right)^{\frac{1}{2}}, \quad \beta_k^{\pm} = \frac{1}{2}(a^{\pm 2}(\tau + l_s^2 k) + a^{\pm 2}(\tau - l_s^2 k)), \end{aligned} \quad (13)$$

here l_s is the string length scale, $z = a\dot{\phi}/H$, $\mathcal{R}_k = u_k(\tilde{\eta})/z_k(\tilde{\eta})$, $k_0 = (\beta_k^+/\beta_k^-)^{1/4}l_s$ and the prime denotes derivative with respect to the modified conformal time $\tilde{\eta}$. Thus the equation of the scalar perturbation can be written as

$$u_k'' + \left(k^2 - \frac{z_k''}{z_k} \right) u_k = 0. \quad (14)$$

After a lengthy but straightforward calculation, we get (up to the first order of the slow-roll parameters and μ defined as below to describe the spacetime noncommutative affects)

$$\frac{z_k''}{z_k} = 2(aH)^2 \left(1 + \frac{5}{2}\epsilon - \frac{3}{2}\eta - 2\mu \right), \quad (15)$$

where $\mu = H^2 k^2/(a^2 M_s^4)$, k is the comoving fourier mode and $M_s = l_s^{-1}$ is the string mass scale. In this paper we only consider the case in which the perturbations are generated inside the horizon. For the slow-roll inflation, the conformal time χ can be approximately integrated out from equation (8),

$$\chi = \int \frac{dt}{a} = \int \frac{da}{a^2 H} \approx \frac{-1}{aH}(1 + \epsilon). \quad (16)$$

And from the third equation in (13), we get

$$\chi \simeq (1 + \mu)^{-1} \tilde{\eta}, \quad (17)$$

therefore

$$aH \simeq \frac{-1}{\chi}(1 + \epsilon) \simeq \frac{-1}{\tilde{\eta}}(1 + \epsilon + \mu). \quad (18)$$

Using equation (15) and (18), we obtain from equation (14)

$$u_k'' + \left(k^2 - \frac{1}{\tilde{\eta}^2} \left(\nu^2 - \frac{1}{4} \right) \right) u_k = 0, \quad (19)$$

where $\nu = \frac{3}{2} + 3\epsilon - \eta$. We notice that this equation is similar to the commutative case and the only difference is that the conformal time χ is replaced by the modified conformal time $\tilde{\eta}$. Next, we choose the initial conditions

$$u_k = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}. \quad (20)$$

The solution of equation (19) is

$$u_k = \frac{1}{2} \sqrt{\pi} e^{i(\nu + \frac{1}{2})\pi/2} (-\tilde{\eta})^{1/2} H_\nu^{(1)}(-k\tilde{\eta}), \quad (21)$$

where $H_\nu^{(1)}$ is the Hankel's function of the first kind. At the superhorizon scales ($k^2 \ll \frac{z_k''}{z_k}$), the solution can be expressed as

$$u_k \simeq \frac{1}{\sqrt{2k}} (-k\tilde{\eta})^{\frac{1}{2}-\nu} \simeq \frac{1}{\sqrt{2k}} \left(\frac{k}{aH} \right)^{\frac{1}{2}-\nu} (1+\mu)^{\frac{1}{2}-\nu}. \quad (22)$$

After a simple calculation, we have, from the second and forth equation of (13),

$$y_k \simeq 1 + \frac{H^2 k^2}{a^2 M_s^4} = 1 + \mu. \quad (23)$$

Thus the power spectrum on superhorizon scales of the comoving curvature can be expressed as

$$P_s \simeq \frac{k^3}{2\pi^2} \left| \frac{u_k}{z_k(\tilde{\eta})} \right|^2 \simeq \frac{1}{2\epsilon} \frac{1}{M_p^2} \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{2\eta-6\epsilon} (1+\mu)^{-4-6\epsilon+2\eta}, \quad (24)$$

here $\tilde{\eta}$ is the time when the fluctuation mode k crosses the Hubble radius ($z_k''/z_k = k^2$). Plugging this condition into equation (15), we get

$$k^2 = 2(aH)^2 \left(1 + \frac{5}{2}\epsilon - \frac{3}{2}\eta - 2\mu \right). \quad (25)$$

Or perturbatively up to the first of the slow-roll parameter ϵ , η and the noncommutative parameter μ , we get

$$d \ln k = (1 - \epsilon + 4\epsilon\mu) H dt, \quad (26)$$

and

$$\frac{d\mu}{d \ln k} = (1 + \epsilon - 4\epsilon\mu) \frac{1}{H} \frac{d}{dt} \left(\frac{H^2 k^2}{a^2 M_s^4} \right) \simeq -4\epsilon\mu. \quad (27)$$

Using equation (24), (26) and (27), we find the spectral index of the scalar perturbation and its running

$$n_s - 1 \equiv s = \frac{d \ln P_s}{d \ln k} = -6\epsilon + 2\eta + 16\epsilon\mu, \quad (28)$$

$$\frac{dn_s}{d \ln k} = -24\epsilon^2 + 16\epsilon\eta - 2\xi^2 - 32\epsilon\eta\mu. \quad (29)$$

When $l_s \rightarrow 0$ or $M_s \rightarrow +\infty$, the noncommutative parameter $\mu = H^2 k^2 / (a^2 M_s^4) \rightarrow 0$, equation (28) and (29) reproduces the results in commutative case.

Inflation predicts that there are also tensor perturbations of the metric. Because the amplitude of primordial tensor perturbations of the metric during the period of inflation only depends on the energy scale of the inflation, we can determine the energy scale of a model by the measurement of the amplitude of the primordial gravitational wave perturbations. In general the linear tensor perturbations may be written as

$$ds^2 = a^2(\chi) \left(d\chi^2 - (\delta_{ij} + 2h_{ij}) dx^i dx^j \right), \quad (30)$$

where $|h_{ij}| \ll 1$. The tensor h_{ij} has six degrees of freedom, but the tensor fluctuations are traceless and transverse. With these four constraints, there remain two physical degrees of freedom, or polarizations. Notice that tensors h_{ij} are gauge-invariant and therefore represent two physical degrees of freedom. Here the stress-energy momentum tensor is diagonal, as the one provided by the inflation potential. Thus the tensor mode does not have a source in its equation of motion and then, in the commutative case, the action of the tensor mode is simply

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} \partial_\sigma h_{ij} \partial^\sigma h_{ij}, \quad (31)$$

similar to the action of two independent massless scalar fields. Using the same argument as in the scalar perturbations in [12], in the noncommutative spacetime background, the gauge-invariant tensor amplitude can be expressed as

$$h_k = \frac{v_k}{a_k M_p}, \quad (32)$$

where $a_k = ay_k$, where y_k is given in equation (23). The motion equation of v_k becomes

$$v_k'' + \left(k^2 - \frac{(a_k)''}{a_k} \right) v_k = 0. \quad (33)$$

After a lengthy calculation, we find

$$\frac{(a_k)''}{a_k} = 2(aH)^2 \left(1 - \frac{\epsilon}{2} - 2\mu \right). \quad (34)$$

and

$$v_k'' + \left(k^2 - \frac{1}{\tilde{\eta}^2} \left(\nu^2 - \frac{1}{4} \right) \right) v_k = 0, \quad (35)$$

where $\nu = 3/2 + \epsilon$. Similar to the scalar fluctuation, the tensor modes at superhorizon scales can be solved

$$v_k \simeq \frac{1}{\sqrt{2k}} \left(\frac{k}{aH} \right)^{\frac{1}{2}-\nu} (1 + \mu)^{\frac{1}{2}-\nu}. \quad (36)$$

In the end, the amplitude of the metric tensor perturbations is found to be

$$P_T = 2 \times \frac{k^3}{2\pi^2} |h_k|^2 = 2 \times \frac{k^3}{2\pi^2} \frac{1}{M_p^2} \left| \frac{v_k}{a_k} \right|^2 = 2 \times \frac{1}{M_p^2} \left(\frac{H}{2\pi} \right)^2 \left(\frac{k}{aH} \right)^{-2\epsilon} (1 + \mu)^{-4-2\epsilon}, \quad (37)$$

where the factor 2 comes from the number of the independent physical degrees of freedom of the tensor perturbation h_{ij} . Using eqs. (24) and (37), we find

$$P_T \simeq 4\epsilon P_s. \quad (38)$$

In power-law inflation $\epsilon = 1/p$ where p is the power of the time in the scale factor of the universe, thus $P_T = (4/p)P_s$, the same as the result in [4]. The spectral index n_T of the tensor perturbations and its running $dn_T/d\ln k$ are

$$n_T = \frac{d \ln P_T}{d \ln k} = -2\epsilon + 16\epsilon\mu, \quad (39)$$

and

$$\frac{dn_T}{d \ln k} = -8\epsilon^2 + 4\epsilon\eta - 32\epsilon\eta\mu. \quad (40)$$

These two equations also reproduce the results in commutative case when $\mu \rightarrow 0$. The spectrum of the tensor perturbations in commutative case must be red ($n_T < 1$), since the slow-roll parameter ϵ is positive. In noncommutative spacetime, it can be blue when $\mu > 1/8$. We find that the amplitude of the tensor perturbations are also suppressed in noncommutative spacetime.

From equation (24) and (37), we see that the spacetime noncommutative effects suppress the power spectrum of the primordial scalar and tensor perturbations approximately with a same factor $(1 + \mu)^{-4}$, leading to a more blue spectrum with a correction $+16\epsilon\mu$ appearing in (28) and (39), where ϵ must be positive by definition in (4). Using the argument in [12], we find that the only difference in the noncommutative case from the commutative case is replacing the conformal time χ with modified conformal time $\tilde{\eta}$ and $\chi \simeq (1 + \mu)^{-1}\tilde{\eta}$ in equation (17). And $\tilde{\eta}$ plays the role of the conformal time in the commutative case. Thus we can get the noncommutative result by replacing χ in the commutative case by $(1 + \mu)^{-1}\chi$ which means a delay of the time when the fluctuation mode cross outside the horizon, since both of the conformal and the modified conformal time are negative and $(1 + \mu)^{-1} < 1$. After a simple calculation, we find that the Hubble constant becomes smaller in the noncommutative case than in the commutative case by a factor $(1 + \mu)^{-1}$.

Since the power spectrum $P \sim H^2/(M_p y_k)^2$ and $y_k = 1 + \mu$ in eq.(23), we predict that the power spectrum in the noncommutative case is suppressed by a factor $(1 + \mu)^{-4}$.

We also note that in a de Sitter space, the noncommutative inflation is the same as the commutative one, since $\epsilon = 0$. The reason is that the Hubble constant is not varying with time and there are no time delaying effects due to the noncommutative effects.

Before we discuss how to match the results of the WMAP group [1], we briefly review their results. For the scalar modes, the spectral index and its running at two different scales are

$$\begin{aligned} n_s &= 0.93 \pm 0.03, \quad \frac{dn_s}{d \ln k} = -0.031^{+0.016}_{-0.017} \quad \text{at} \quad k = 0.05 \text{Mpc}^{-1}, \\ n_s &= 1.20^{+0.12}_{-0.11}, \quad \frac{dn_s}{d \ln k} = -0.077^{+0.050}_{-0.052} \quad \text{at} \quad k = 0.002 \text{Mpc}^{-1}. \end{aligned} \quad (41)$$

Here we use the new results revised by Peiris et al. on 12, May. The data of the WMAP give rise to a maximum of the tensor/scalar ratio leading to a constraint on the slow-roll parameter $\epsilon < 1.28/16 = 0.08$ (95%CL).

If we want to directly fit the data of WMAP by using the inflation model in noncommutative spacetime, we need to calculate the primordial power spectrum exactly, use some sophisticated techniques, such as running CMBfast, to compute the C_l^{TT} etc and compare them with the data directly with taking the likelihood into account. In our paper, we want to show that we can produce the large enough running of the spectral index of the power spectrum in the best-fit results of WMAP group.

We have showed that the spacetime noncommutative power-law inflation can fit the spectral index and its running quite nicely in [3] and [5]. Here we extend our previous discussions to a general slow-roll inflation model and discuss how to match the index of the power spectrum and its running by using our previous results. From equation (28), we have $\eta = \frac{1}{2}s + 3\epsilon - 8\epsilon\mu$. With this substituted into (29), it becomes

$$\frac{dn_s}{d \ln k} = s^2 + (13 - 48\mu)s\epsilon + (28 - 304\mu + 512\mu^2)\epsilon^2 - 2\zeta^2. \quad (42)$$

If spacetime is commutative ($\mu = 0$), equation (42) becomes

$$\frac{dn_s}{d \ln k} = s^2 + 13s\epsilon + 28\epsilon^2 - 2\zeta^2. \quad (43)$$

Thus in commutative case, ζ^2 must be very large in order to get a large enough negative value of $dn_s/d \ln k$ in (43), specially when the CMB power spectrum is blue ($s > 0$), to fit

the WMAP data, since $\epsilon > 0$. However it is quite hard to get a large ζ^2 in the known typical inflation model [13] and has been also discussed in [14]. If we take the noncommutative effects into account, the second or the third term of equation (42) will become negative for some suitable values of μ , there is no longer the need of a large ζ^2 . In practice, we shall ignore ζ^2 and show the constraint on values of ϵ and of μ in order to fit the experimental data, in fig. 1.

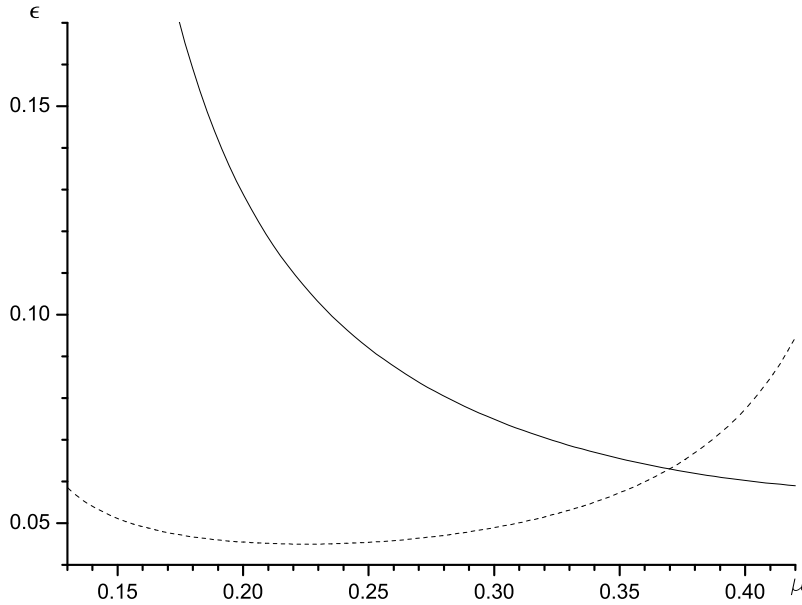


Figure 1. The value of the solid line is $n_s = 1.20$, $dn_s/d\ln k = -0.077$ and the dash line is $n_s = 0.93$, $dn_s/d\ln k = -0.031$ by using eqs.(28) and (42), here we neglect ζ^2 .

Further, the constraints on the parameters ϵ and μ are loosened after we take the contribution of ζ^2 into consideration. We see from fig.1 that it is possible to naturally realize a large enough running of spectral index of the scalar metric perturbations after we take the spacetime noncommutative effects into consideration.

In the following we shall check some typical inflation models. According to equation (28) (29) (39) and (40), we see that the noncommutative term μ always appears in a product with the slow-roll parameter ϵ . Thus, there is a significant effect in the models only when ϵ is large. We will see that the noncommutative effects in cases of the power-law model and the chaotic model are large enough to realize the running of the index of the power spectrum.

Case 1.

The potential $V = \lambda^4 \exp(-\sqrt{2/p}(\phi/M_p))$ leads to power-law inflation with $a \sim t^p$ and the slow-roll parameters are $\epsilon = 1/p$, $\eta = 2/p$ and $\xi^2 = 4/p^2$. Since ϵ is a constant, we can integrate equation (27) to get

$$\frac{d\mu}{d \ln k} = -4\epsilon\mu = -\frac{4}{p}\mu, \quad (44)$$

and

$$\mu = \left(\frac{k}{k_c}\right)^{-4\epsilon} = \left(\frac{k}{k_c}\right)^{-4/p}. \quad (45)$$

Equations (28) and (29) become

$$n_s = 1 - \frac{2}{p} + \frac{16}{p}\mu = -\frac{2}{p} + \frac{16}{p} \left(\frac{k}{k_c}\right)^{-4/p}, \quad (46)$$

$$\frac{dn_s}{d \ln k} = -\frac{64}{p^2}\mu = -\frac{64}{p^2} \left(\frac{k}{k_c}\right)^{-4/p}. \quad (47)$$

These formulas are exactly the same as in [3].

Case 2.

Potential $V = \lambda^4(\phi/M)^p$ ($p \geq 2$). The number of e-folds is $N \simeq \frac{1}{M_p^2} \int_{\phi_{end}}^{\phi_N} \frac{V}{V'} d\phi \simeq \phi_N^2/(2pM_p^2)$, so $\phi_N = \sqrt{2pN}M_p$. The slow-roll parameters are $\epsilon = (p^2/2)M_p^2/\phi^2 = p/(4N)$, $\eta = (p-1)/(2N)$ and $\xi^2 = (p-1)(p-2)/(4N^2)$. Thus the spectral index and its running become

$$n_s - 1 = s = \frac{1}{N} \left(-1 - \frac{1}{2}(1 - 8\mu)p \right), \quad (48)$$

and

$$\frac{dn_s}{d \ln k} = -\frac{1}{N^2} \left(1 + \frac{p}{2} + 4p(p-1)\mu \right). \quad (49)$$

For $N = 50$ and $p = 2$, $n_s = 0.96$ and $dn_s/d \ln k = -0.0008$, the running is too small to fit the experimental data in the commutative case ($\mu = 0$). Eq. (48) also tells us that the power spectrum must be red in this case. The situation changes in the noncommutative spacetime background. When $\mu > (1 + 2/p)/8$, the spectrum becomes blue. For a small k , the larger μ the earlier it crosses outside the horizon and thus produces a larger spectral index. First, we try to fit the data at $k = 0.05 Mpc^{-1}$ in eq. (41). We show the fitting parameters in fig. 2 - 4.

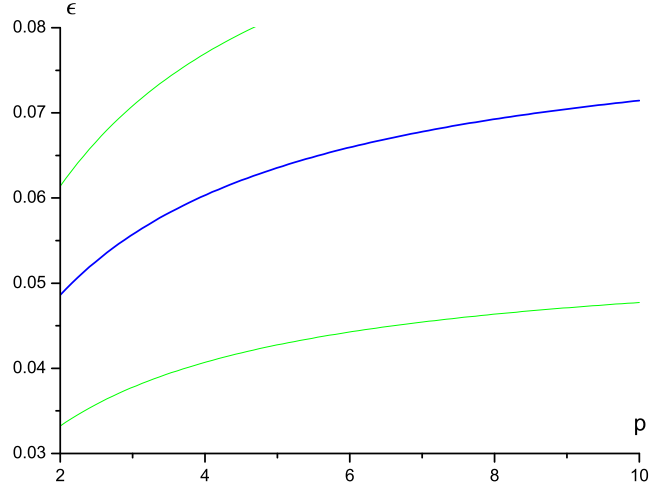


Figure 2. where ϵ is the slow-roll parameter with the constraints $\epsilon < 0.08$ from WMAP and p is the index of the inflaton in the potential.

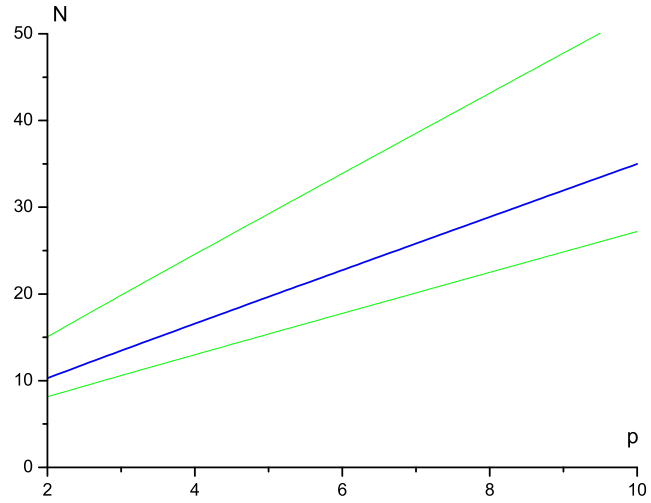


Figure 3. where N is the e -folding number and p is the index of the inflaton in the potential.

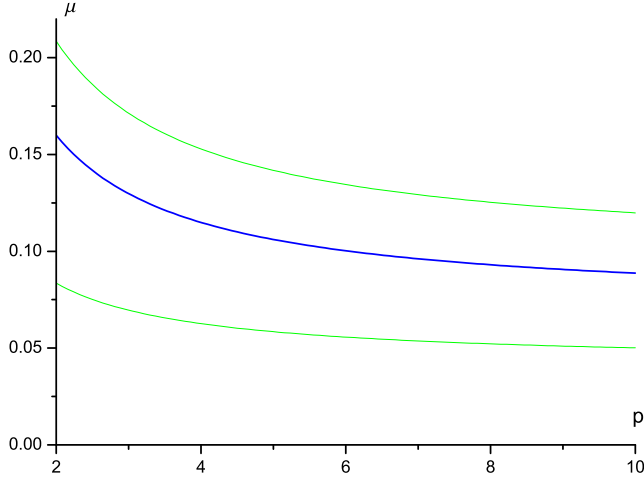


Figure 4. where $\mu = H^2 k^2 / (a^2 M_s^4)$ is the noncommutative parameter and p is the index of the inflaton in the potential.

The blue lines in fig. 2 - 4 correspond to $n_s = 0.93$ and $dn_s/d \ln k = -0.031$ and the green lines show the range of the parameters to fit the likelihood of n_s and $dn_s/d \ln k$. In general, the larger the slow-roll parameter ϵ , the larger the noncommutative effects. From fig. 2-4, we choose $p = 10$ as an example in order to get a large e-folds number. According to the blue lines in these three figures, we can get $N = 35$, $\epsilon = 0.071$ and $\mu = 0.089$. Since $\mu = 0.089 \ll 1$, we can trust our perturbative results. We can also fit the data at the mode $k = 0.002 Mpc^{-1}$ and the results are $N = 42$, $\epsilon = 0.060$ and $\mu = 0.36$. We can see that the spectrum runs from the blue one ($n_s = 1.20$) at $N = 42$ to the red one ($n_s = 0.93$) at $N = 35$. This example shows us that the chaotic inflation model can fit the results of WMAP in noncommutative spacetime.

Case 3.

Potential $V = \lambda^4 [1 - (\frac{\phi}{M})^p]$ ($p \geq 2$). For $p = 2$, the number of e-folds at ϕ_N before the end of inflation is given by $N = \int_t^{t_{end}} H dt \simeq \frac{1}{M_p^2} \int_{\phi_{end}}^{\phi_N} \frac{V}{V'} d\phi \simeq (M^2/2M_p^2) \ln(M/\phi_N)$, or $\phi_N \simeq M \exp(-2NM_p^2/M^2)$, where we put $\phi_{end} \simeq M$. The slow-roll parameters are $\epsilon = 2M_p^2 \phi_N^2 / M^4 = 2(M_p^2/M^2) \exp(-4N(M_p^2/M^2))$, $\eta = -2(M_p^2/M^2)$ and $\xi^2 = 0$. In this case, $-\eta \ll 1$ implies $M_p \ll M$. For instance assuming $M = 10M_p$ and $N = 50$, we have $\epsilon = 0.0027$, $\eta = -0.02$, $n_s = 0.96$ and $dn_s/d \ln k = -0.001$. Because ϵ is small, $\eta < 0$ and $-32\epsilon\eta\mu = 0.0017\mu > 0$, The spacetime noncommutative effects can not improve this model. The same is true for $p > 2$.

Case 4.

We consider the P-term inflation [15]. In unit with $M_p = 1$, the potential can be expressed as

$$V = \frac{g^2 \chi^2}{2} \left(1 + \frac{g^2}{8\pi^2} \ln \frac{s^2}{\chi} + \frac{f}{8} s^4 + \dots \right), \quad (50)$$

here $0 \leq f \leq 1$ and special case $f = 1$ corresponds to F-term inflation, $f = 0$ corresponds to D-term inflation. In P-term model the parameter f is the contribution from supergravity which permits a controllable running of the scalar spectral index (no running in D-term case). One should note that g^2 in this model is not necessarily related to the gauge coupling constant in GUTs. We have $g \geq 2 \times 10^{-3}$ and $\chi \simeq 10^{-5}$ [15] (constrained from CMB data), which is a natural assumption (if $g < 2 \times 10^{-3}$, one has exactly flat spectrum of density perturbations $n_s = 1$. So we do not consider this case.). Now we check whether this model can fit the experimental data in noncommutative spacetime. For $g \geq 2 \times 10^{-3}$, the inflation is driven by the first term in (50) and we have

$$\epsilon = \frac{1}{2} \left(\frac{g^2}{4\pi^2 s} + \frac{f s^3}{2} \right)^2, \quad (51)$$

$$\eta = -\frac{g^2}{4\pi^2 s^2} + \frac{3f s^2}{2}, \quad (52)$$

$$\xi^2 = \left(\frac{g^2}{2\pi^2 s^3} + 3f s \right) \left(\frac{g^2}{4\pi^2 s} + \frac{f s^3}{2} \right), \quad (53)$$

and $s_N^2 = \frac{g^2 N}{2\pi^2}$ where N is the number of e-folds. According to [15], for $g > 0.15$, inflation in this model is too short, whereas for $g < 0.06$ one can ignore the supergravity term $f s^4/8$ for the description of the last 60 e-folds of inflation. We expect that the spectral index n runs from $n < 1$ at small wavelengths to $n > 1$ at large wavelengths in the intermediate regime $0.06 < g < 0.15$. For example, we show the spectral index and its running in the P-term inflation model in figs. 5 and 6, where we choose $f = 0.8$ and $g = 0.08$.

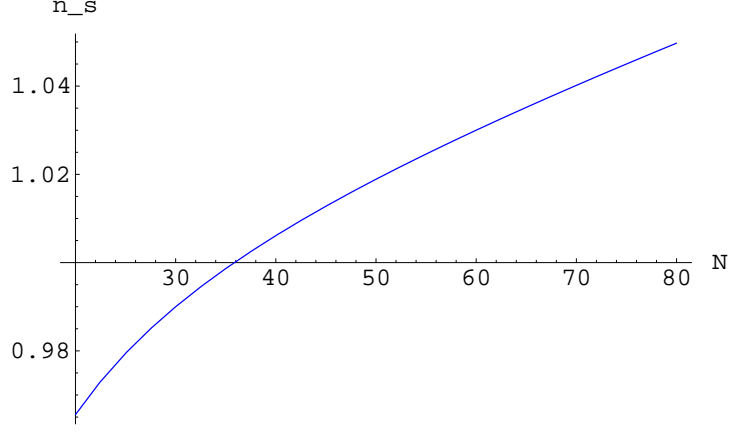


Figure 5. where n_s is the spectral index and N is the number of e -folds.

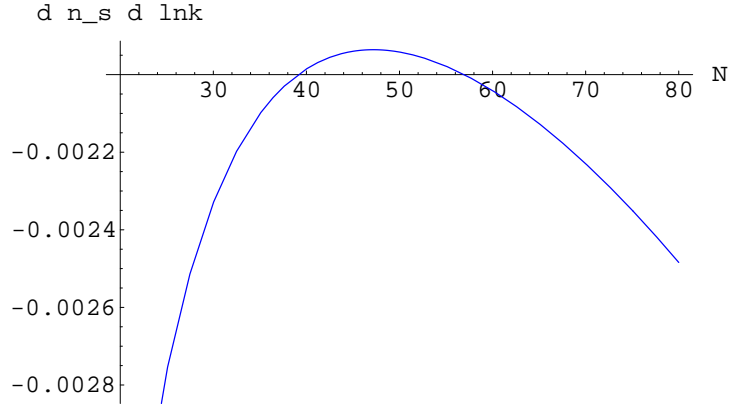


Figure 6. where $dn_s/d\ln k$ is the running of the spectral index and N is the number of e -folds.

From fig. 5 and 6, we see that P-term inflation can not provide a large enough running of the spectral index to fit the experiment data. A similar result is obtained in [16]. And also since $\epsilon = 2 \times 10^{-6} \ll 1$, $\eta = 0.0036$ for $N = 40$ and then $32\epsilon\eta\mu = 7.2 \times 10^{-9}\mu$ which can be ignored, we cannot expect the noncommutative effects improve this model to fit the experiment data also.

To summarize, if we introduce the spacetime uncertainty relations into cosmology, the classical evolution of the inflaton and the universe is not different from the commutative case, since the classical inflaton is homogeneous and the universe is spatially isotropic and homogeneous. However this uncertainty relation leads to the nonlocal coupling in time between the background and the fluctuations. The time when the fluctuation mode crosses the Hubble horizon is delayed for a smaller Hubble constant. Thus the fluctuations

will be smaller than those in the absence of the uncertainty relation. This suppression implies that the spectral index is larger than the one in the commutative case for a long wave-length. With the expansion of the universe, the cosmic scale becomes larger and larger and the effects of the uncertainty relation become smaller and smaller. So the fluctuations of the ultraviolet modes are not different from the prediction of the standard theory.

The observations of WMAP brings about some more radical suggestions, namely a running spectral index of the scalar perturbation, making a transition from $n > 1$ on large scales to $n < 1$ on small scales, and anomalously low quadrupole and octupole. These two results are not anticipated in the usual inflation models. In the last few months, many authors have extensively discussed these results in [3-5,13,16-18]. The results of the cosmological observations are still awaiting for further confirmation. In particular, the future precise observations, for example measurements of C_l^{TE} [19], may be needed to show whether the anomalously low quadrupole and octupole can be trusted or not. We also expect that the future more precise measurements will provide a testing ground for whether spacetime uncertainty is a viable physical model, and for other new physics in the trans-Planckian regime.

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